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SHAKEDOWN OF BARS SUBJECTED TO CYCLES OF LOADS AND TEMPERATURE

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Abstract—A new mathematical method is developed to study the shakedown of bar systems subjected to cycles of loads and temperature allowing for temperature dependence of the yield stresses. For a parallel bar system constrained to have equal displacements at the ends, the shakedown factor is obtained in an explicit analytical form. The possible inadaptation modes of incremental collapse (structural ratchetting) and alternating plasticity (low cycle fatigue) on the boundary of the shakedown domain are determined.

1. INTRODUCTION

The investigations of the shakedown of structures under variable temperature fields were initiated by Prager (1957) who suitably generalized the Melan statical theorem. The generalization of the Koiter kinematical theorem was given by Rozenblum (1965). Prager was also the first to notice that the statement of independence of the load-carrying capacity of structures from the self-equilibrated residual stresses was no longer valid in the case of cyclic thermal stresses. A temperature field, apart from generating certain stress and strain states in a structure, may exert an influence on the mechanical properties of the materials. For the simplest structures subjected to variable temperature and constant loads, solutions were obtained and presented in the form of a Bree diagram (Bree, 1967; Mulcahy, 1976). Gokhfeld and Cherniavski (1980), Ponter and Karadeniz (1985) and König (1987) studied the special mode of perfect incremental collapse of structures subjected to cycles of loads and temperature, applying the upper bound kinematical theorem.

In this study the difficulty of application of the Koiter kinematical theorem is overcome to yield a practical form of the shakedown factor for bar systems subjected to cycles of loads and temperature. The temperature dependence of the yield stress is taken into account while the weak influence of the temperature on the elastic moduli is neglected. In the temperature ranges considered, viscous properties are not pronounced so the time independent framework of the shakedown theory is preserved.

2. APPLICATION OF THE KINEMATICAL THEOREM

Let us consider a system of n bars of constant cross-sections F_i, lengths l_i and volumes $V_i = F_i l_i$ $(i = 1, \dots, n)$ subjected to variable external loads and kinematic constraints in a changing (homogeneous in each bar) temperature field $\theta_i(t)$. The bars are generally made from different materials with the yield stresses:

$$\sigma_{Y_i}(\theta) = \sigma^0_{Y_i}[1 - g_i(\theta)], \tag{1}$$

where $g_i(\theta)$ are some functions of temperature, $g_i(\theta) < 1, g_i(0) = 0$; $\sigma^0_{Y_i}$ are the yield stresses at the reference temperature $\theta = 0$.

Koiter's inadaptation condition (Koiter, 1963; Rozenblum, 1965; König, 1987) for axially loaded bars in the presence of thermal effects, taking into account (1), could be given as:

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$$\sum_i V_i \cdot \int_0^T (\sigma_i^e \, e_i^P + g_i \sigma_{Y_i}^0 |e_i^P|) \, dt \geqslant \sum_i V_i \cdot \int_0^T \sigma_{Y_i}^0 |e_i^P| \, dt, \tag{2}$$

where

$$\varepsilon_i^P = \int_0^T e_i^P \, dt \tag{3}$$

is compatible (i.e. satisfied kinematic constraints), $e_i^P(t)$ is the plastic strain rate; ε_i^P is the plastic strain increment over a cycle; $\sigma_i^e(t)$ is the fictitious elastic stress response of the bars to external loads and temperature field. The index i under the sign of sum runs from 1 to n.

From (2), we determine the shakedown factor k_s as

$$k_s^{-1} = \sup_{e_i^P \in (3)} \left[\sum_i V_i \cdot \int_0^T (\sigma_i^e \, e_i^P + g_i \sigma_{Y_i}^0 |e_i^P|) \, dt \right] \cdot \left[\sum_i V_i \cdot \int_0^T \sigma_{Y_i}^0 |e_i^P| \, dt \right]^{-1}, \tag{4}$$

where $e_i^P \in (3)$ means that the suprimum is taken over all fields e_i^P satisfying condition (3). We will not try to give the shakedown factor defined in (4) a practical meaning, particularly when $g_i(\theta) \neq 0$, but the equation $k_s = 1$ yields the exact shakedown domain boundary in the space of external loads and temperature and when $k_s > 1$ the structure will shakedown and with $k_s < 1$ it fails.

Plastic strain rate fields $e_i^P(t)$ satisfying (3) could be expressed as

$$e_i^P(t) = \begin{cases} \Lambda_i(t) \cdot \varepsilon_i^P, & \int_0^T \Lambda_i \, dt = 1, \quad i \in C_P = \{i | \varepsilon_i^P \neq 0\}, \\ \Lambda_i(t) \cdot \bar{\varepsilon}_i, & \int_0^T \Lambda_i \, dt = 0, \quad i \in C_0 = \{i | \varepsilon_i^P = 0\}, \end{cases} \tag{5}$$

where

$$\varepsilon_i^P \text{ is compatible while } \bar{\varepsilon}_i \text{ is arbitrary.} \tag{6}$$

Our purpose is to transform (4) into a more practical form suitable for applications. Denote:

$$S_i = \int_0^T (|\Lambda_i| - \Lambda_i)/2 \, dt, \tag{7}$$

then from (5) we derive

$$\int_0^T (|\Lambda_i| + \Lambda_i)/2 \, dt = \begin{cases} S_i + 1, & i \in C_P, \\ S_i, & i \in C_0. \end{cases} \tag{8}$$

Denote

$$U_i(t) = \begin{cases} \sigma_i^e(t) \cdot \varepsilon_i^P + g_i(\theta_i(t)) \cdot \sigma_{Y_i}^0 \cdot |\varepsilon_i^P|, & i \in C_P, \\ \sigma_i^e(t) \cdot \bar{\varepsilon}_i + g_i(\theta_i(t)) \cdot \sigma_{Y_i}^0 \cdot |\bar{\varepsilon}_i|, & i \in C_0, \end{cases}$$

$$L_i(t) = \begin{cases} -\sigma_i^e(t) \cdot \varepsilon_i^P + g_i(\theta_i(t)) \cdot \sigma_{Y_i}^0 \cdot |\varepsilon_i^P|, & i \in C_P, \\ -\sigma_i^e(t) \cdot \bar{\varepsilon}_i + g_i(\theta_i(t)) \cdot \sigma_{Y_i}^0 \cdot |\bar{\varepsilon}_i|, & i \in C_0, \end{cases}$$

$$U_i^m = \max_t U_i(t) = U_i(t_{U_i}), \quad L_i^m = \max_t L_i(t) = L_i(t_{L_i}). \tag{9}$$

For $i \in C_p$ we have:

$$\sigma_i^e e_i^p + g_i \sigma_{Y_i}^0 |e_i^p| = \sigma_i^e \varepsilon_i^p \Lambda_i + g_i \sigma_{Y_i}^0 |\varepsilon_i^p||\Lambda_i|$$

$$= \sigma_i^e \varepsilon_i^p \left(\frac{\Lambda_i + |\Lambda_i|}{2} - \frac{|\Lambda_i| - \Lambda_i}{2} \right) + g_i \sigma_{Y_i}^0 |\varepsilon_i^p| \left(\frac{\Lambda_i + |\Lambda_i|}{2} + \frac{|\Lambda_i| - \Lambda_i}{2} \right)$$

$$= U_i \frac{\Lambda_i + |\Lambda_i|}{2} + L_i \frac{|\Lambda_i| - \Lambda_i}{2},$$

$$\int_0^T |e_i^p| \, dt = |\varepsilon_i^p| \cdot \int_0^T |\Lambda_i| \, dt = |\varepsilon_i^p| \cdot \int_0^T \left(\frac{\Lambda_i + |\Lambda_i|}{2} + \frac{|\Lambda_i| - \Lambda_i}{2} \right) dt = |\varepsilon_i^p| (2S_i + 1).$$

Similarly, for $i \in C_0$,

$$\sigma_i^e e_i^p + g_i \sigma_{Y_i}^0 |e_i^p| = U_i \frac{\Lambda_i + |\Lambda_i|}{2} + L_i \frac{|\Lambda_i| - \Lambda_i}{2},$$

$$\int_0^T |e_i^p| \, dt = |\bar{\varepsilon}_i| 2S_i.$$

Now substituting (5) into (4), taking into account the last formulae, we come to

$$k_s^{-1} = \sup_{\varepsilon_i^p \in (6), \bar{\varepsilon}_i} \; \sup_{S_i \geq 0, \Lambda_i \in (7, 8)} \frac{\sum_i V_i \cdot \int_0^T \left[U_i \cdot \frac{|\Lambda_i| + \Lambda_i}{2} + L_i \cdot \frac{|\Lambda_i| - \Lambda_i}{2} \right] dt}{\sum_{i \in C_P} V_i \sigma_{Y_i}^0 (2S_i + 1) |\varepsilon_i^p| + \sum_{i \in C_0} V_i \sigma_{Y_i}^0 2S_i |\bar{\varepsilon}_i|}. \tag{10}$$

Let us consider the integral expression in the numerator of (10)—first for the case $i \in C_P$:

$$\int_0^T \left[U_i \cdot \frac{|\Lambda_i| + \Lambda_i}{2} + L_i \cdot \frac{|\Lambda_i| - \Lambda_i}{2} \right] dt \leq U_i^m \cdot \int_0^T \frac{|\Lambda_i| + \Lambda_i}{2} dt + L_i^m \cdot \int_0^T \frac{|\Lambda_i| - \Lambda_i}{2} dt$$
$$= U_i^m \cdot (S_i + 1) + L_i^m \cdot S_i.$$

On the other hand, taking $\Lambda_i = (S_i + 1) \cdot \delta(t - t_{U_i}) - S_i \cdot \delta(t - t_{L_i})$ satisfying (7), (8) we have: ($\delta(t)$ is the Dirac function)

$$\int_0^T \left[U_i \cdot \frac{|\Lambda_i| + \Lambda_i}{2} + L_i \cdot \frac{|\Lambda_i| - \Lambda_i}{2} \right] dt = U_i^m \cdot (S_i + 1) + L_i^m \cdot S_i.$$

Therefore we conclude

$$\sup_{\Lambda_i \in (7, 8)} \int_0^T \left[U_i \cdot \frac{|\Lambda_i| + \Lambda_i}{2} + L_i \cdot \frac{|\Lambda_i| - \Lambda_i}{2} \right] dt = U_i^m \cdot (S_i + 1) + L_i^m \cdot S_i \quad (i \in C_P).$$

Similarly, for $i \in C_0$ one obtains

$$\sup_{\Lambda_i \in (7, 8)} \int_0^T \left[U_i \cdot \frac{|\Lambda_i| + \Lambda_i}{2} + L_i \cdot \frac{|\Lambda_i| - \Lambda_i}{2} \right] dt = U_i^m \cdot S_i + L_i^m \cdot S_i.$$

Thus (10) becomes

$$k_s^{-1} = \sup_{\varepsilon_i^{\mathrm{P}} \in (6), \bar{\varepsilon}_i} \sup_{S_i \geqslant 0} \frac{\sum_{i \in C_{\mathrm{P}}} V_i [U_i^m (S_i + 1) + L_i^m \cdot S_i] + \sum_{i \in C_0} V_i (U_i^m + L_i^m) S_i}{\sum_{i \in C_{\mathrm{P}}} V_i \sigma_{Y_i}^0 (2S_i + 1) |\varepsilon_i^{\mathrm{P}}| + \sum_{i \in C_0} V_i \sigma_{Y_i}^0 2 S_i |\bar{\varepsilon}_i|}. \tag{11}$$

Denote

$$\hat{S}_i = \begin{cases} 2S_i \cdot \sigma_{Y_i}^0 \cdot |\varepsilon_i^{\mathrm{P}}| \cdot V_i, & i \in C_{\mathrm{P}}, \\ 2S_i \cdot \sigma_{Y_i}^0 \cdot |\bar{\varepsilon}_i| \cdot V_i, & i \in C_0, \end{cases}$$

$$X = \sum_i \hat{S}_i, \tag{12}$$

then (11) could be rewritten as

$$k_s^{-1} = \sup_{\varepsilon_i^{\mathrm{P}} \in (6), \bar{\varepsilon}_i} \sup_{\hat{S}_i \in (12), X \geqslant 0} \frac{\sum_{i \in C_{\mathrm{P}}} \Big/ \left[V_i U_i^m + \frac{U_i^m + L_i^m}{2\sigma_{Y_i}^0 |\varepsilon_i^{\mathrm{P}}|} \hat{S}_i \right] + \sum_{i \in C_0} \frac{U_i^m + L_i^m}{2\sigma_{Y_i}^0 |\bar{\varepsilon}_i|} \hat{S}_i}{X + \sum_{i \in C_{\mathrm{P}}} V_i \cdot \sigma_{Y_i}^0 \cdot |\varepsilon_i^{\mathrm{P}}|}. \tag{13}$$

Denote

$$W = \max_{C_0, C_{\mathrm{P}}} \left\{ \max_{i \in C_{\mathrm{P}}} \frac{U_i^m + L_i^m}{2\sigma_{Y_i}^0 |\varepsilon_i^{\mathrm{P}}|}, \quad \max_{i \in C_0} \frac{U_i^m + L_i^m}{2\sigma_{Y_i}^0 |\bar{\varepsilon}_i|} \right\}, \tag{14}$$

and let j be the point where the maximum is reached.

It is easy to see that

$$k_s^{-1} \leqslant \sup_{\varepsilon_i^{\mathrm{P}} \in (6), \bar{\varepsilon}_i} \sup_{X \geqslant 0} \frac{\sum_{i \in C_{\mathrm{P}}} V_i \cdot U_i^m + X \cdot W}{X + \sum_{i \in C_{\mathrm{P}}} V_i \cdot \sigma_{Y_i}^0 \cdot |\varepsilon_i^{\mathrm{P}}|}. \tag{15}$$

On the other hand, putting $\hat{S}_j = X$, $\hat{S}_i = 0$ $(i \neq j)$ satisfying (12) into (13), taking into account (14), we deduce

$$k_s^{-1} \geqslant \sup_{\varepsilon_i^{\mathrm{P}} \in (6), \bar{\varepsilon}_i} \sup_{X \geqslant 0} \frac{\sum_{i \in C_{\mathrm{P}}} V_i \cdot U_i^m + X \cdot W}{X + \sum_{i \in C_{\mathrm{P}}} V_i \cdot \sigma_{Y_i}^0 \cdot |\varepsilon_i^{\mathrm{P}}|}.$$

This inequality is the opposite of (15), therefore the right-hand expression is the exact value of k_s^{-1}. Furthermore the expression after sup monotonously depends on $X \in [0, +\infty)$, so the suprimum is attained at $X = 0$ or $X = +\infty$:

$$k_s^{-1} = \sup_{\varepsilon_i^{\mathrm{P}} \in (6), \bar{\varepsilon}_i} \max \left\{ \frac{\sum_{i \in C_{\mathrm{P}}} V_i \cdot U_i^m}{\sum_{i \in C_{\mathrm{P}}} V_i \cdot \sigma_{Y_i}^0 \cdot |\varepsilon_i^{\mathrm{P}}|}, \quad W \right\} = \max\{I, A\}, \tag{16}$$

where

$$I = \sup_{\varepsilon_i^P \in (6)} \frac{\sum_{i \in C_P} V_i \cdot U_i^m}{\sum_{i \in C_P} V_i \cdot \sigma_{Y_i}^0 \cdot |\varepsilon_i^P|} = \sup_{\varepsilon_i^P \in (6), t_i} \frac{\sum_i V_i [\sigma_i^e(t_i)\varepsilon_i^P + g_i(\theta_i(t_i))\sigma_{Y_i}^0 |\varepsilon_i^P|]}{\sum_i V_i \cdot \sigma_{Y_i}^0 \cdot |\varepsilon_i^P|}, \tag{17}$$

$$A = \sup_{\varepsilon_i^P \in (6), \bar{\varepsilon}_i} W = \sup_{\bar{\varepsilon}_i, i} \frac{U_i^m + L_i^m}{2\sigma_{Y_i}^0 |\bar{\varepsilon}_i|} = \max_{i,t,t'} \frac{\sigma_i^e(t) - \sigma_i^e(t') + [g_i(\theta_i(t)) + g_i(\theta_i(t'))] \cdot \sigma_{Y_i}^0}{2\sigma_{Y_i}^0}. \tag{18}$$

Equations (16)–(18) are much simpler than the initial formulation of Koiter's theorem (4) because the time integrals and the plastic strain rate field $\dot{e}_i^P(t)$ have vanished.

Remember that $k_s = 1$ determines the boundary of the shakedown domain. It is interesting to notice that two local modes of nonshakedown from (5) are separated in (16) into the global modes of incremental collapse $\varepsilon_i^P \neq 0$ (17) and alternating plasticity $\bar{\varepsilon}_i \neq 0$ (18). Therefore on the part of the shakedown boundary, where $1 = k_s = I > A$, we have the case of structural ratchetting while on the other part where $1 = k_s = A > I$, the alternating plasticity mode might be the predominant one (at the same time we should not rule out the possibility of some mixed mode collapse there).

3. A SYSTEM OF PARALLEL BARS

Let us consider a system consisting of n parallel bars constrained to have equal displacements at the right ends, while the other ends are fixed (Fig. 1). The system is subjected to a variable load in a changing temperature field

$$P_L \leqslant P(t) \leqslant P_U, \quad \theta_{i_L} \leqslant \theta_i(t) \leqslant \theta_{i_U}, \quad (i = 1, \ldots, n). \tag{19}$$

Let α_i, E_i denote the linear thermal expansion coefficient and elastic modulus of i-bar and suppose $g_i(\theta) = \beta_i \cdot \theta$, $\beta_i \geqslant 0$ is a material constant.

The kinematic constraint on the strains ε_i of bars imposes

$$\varepsilon_i \cdot l_i = u, \tag{20}$$

where u is the horizontal displacement of the right ends of the bars.

The elastic stress response to the external load and temperature field in the i-bar is deduced from the equilibrium equation of the system and (20):

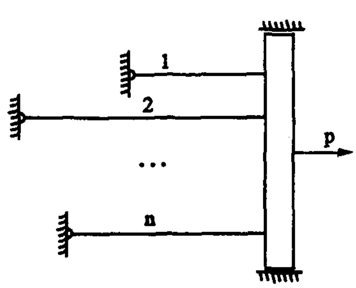

Fig. 1.

$$\sigma_i^e = (\varepsilon_i - \alpha_i\theta_i)E_i = E_i \cdot \left(P + \sum_j E_jF_j\alpha_j\theta_j\right) / \left(l_i \cdot \sum_j E_jF_j/l_j\right) - \alpha_iE_i\theta_i.$$

Because of the independence of the external load from the temperature field in the bars [see eqn (19)] and linear dependence of the expressions after sup and max in (17) and (18) on P and θ_i in our case, one can calculate (17) and (18) easily:

$$I = \left(\sum_i F_i\sigma_{Y_i}^0\right)^{-1} \cdot \max_{P=P_u,P_L;\theta_i=\theta_{iU},\theta_{iL}} \left[\left|\sum_i F_i \cdot \left(E_i \cdot \left(P + \sum_j E_jF_j\alpha_j\theta_j\right) \Big/ \left(l_i \cdot \sum_j E_jF_j/l_j\right) - \alpha_iE_i\theta_i\right)\right| + \sum_i F_i\beta_i\sigma_{Y_i}^0\theta_i\right], \quad (21)$$

$$A = \max_i (2\sigma_{Y_i}^0)^{-1} \cdot \left[E_i \cdot \left(P_U - P_L + \sum_{j \neq i} E_jF_j\alpha_j \cdot (\theta_{U_j} - \theta_{L_j})\right) \Big/ \left(l_i \cdot \sum_j E_jF_j/l_j\right) + \left(1 - E_iF_i \Big/ \left(l_i \cdot \sum_j E_jF_j/l_j\right)\right) \cdot E_i\alpha_i(\theta_{U_i} - \theta_{L_i}) + \sigma_{Y_i}^0\beta_i(\theta_{U_i} + \theta_{L_i})\right]. \quad (22)$$

Thus (16) is resolved in an explicit form, which gives the shakedown domain as well as the possible modes of inadaptation. Various special cases could be considered but we restrict ourselves to a few illustrative examples.

Example 1: For two bars of the same material and of the same constant cross-section F with the lengths l and $2l$; $P_L = -2P_U/3$, $P_U > 0$, $\theta_1 \equiv \theta_2 \equiv 0$; (16), (21), (22) are reduced to the same result as the one obtained by the static approach [see König (1987)]:

$$k_s^{-1} = \max\{I_1, A_1\} = \max\{P_U/(2F\sigma_Y^0), \quad 5P_U/(9F\sigma_Y^0)\} = 5P_U/(9F\sigma_Y^0),$$

and we have the case of alternating plasticity at inadaptation.

Example 2: Two bars of the same material and same length l with the cross-sections $F_1 = F, F_2 = (f-1)\cdot F\ (f \geqslant 2)$; $0 \leqslant P \leqslant P_U$; $0 \leqslant \theta_1 \leqslant \theta_U$; $\theta_2 \equiv 0$. Elementary calculations from formulae (21), (22) give us

$$k_s^{-1} = \max\{I_2, A_2\} = \max\left\{\frac{P_U}{f\cdot F\cdot\sigma_Y^0} + \frac{\beta\cdot\theta_U}{f}, \frac{P_U}{2f\cdot F\cdot\sigma_Y^0} + \left(\frac{f-1}{2f}\cdot\frac{E\cdot\alpha}{\sigma_Y^0} + \beta/2\right)\cdot\theta_U\right\}.$$

In the plane of coordinates θ_U, P_U (Fig. 2), the shakedown domain is given as the area

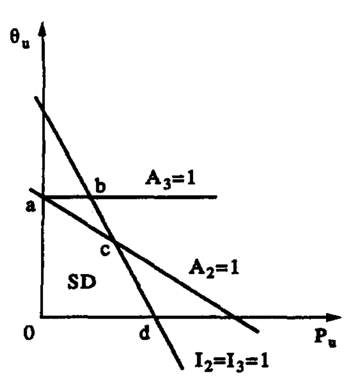

Fig. 2.

inside *oacd* and on *ac* ($A_2 = 1$) we have alternating plasticity while on *cd* ($I_2 = 1$) we have incremental collapse. Taking into account the temperature dependence of the yield stress in this example ($\beta > 0$) leads to a reduction of the shakedown domain.

Example 3: Almost the same data as those given in the previous example; the only change is: the load becomes constant $P = P_U = \text{const.}$ We have

$$k_s^{-1} = \max\{I_3 \cdot A_3\} = \max\left\{\frac{P_U}{f \cdot F \cdot \sigma_Y^0} + \frac{\beta \cdot \theta_U}{f}, \left(\frac{f-1}{2f} \cdot \frac{E \cdot \alpha}{\sigma_Y^0} + \beta/2\right) \cdot \theta_U\right\}.$$

The shakedown domain is increased (in comparison with that of example 2) to domain *oabd* (Fig. 2) and on *ab* ($A_3 = 1$) we have alternating plasticity while on *bd* ($I_3 = 1$)—ratchetting.

Example 4: We come back to the general case (21), (22) and suggest that

$$P_L \leqslant P(t) \leqslant P_U; \quad \theta_i(t) \equiv \theta_{i_U} = \text{const.}, \quad i = 1, \ldots, n.$$

We have:

$$k_s^{-1} = \max\{I, A\} = \max\left\{\frac{\max(P_U, -P_L) + \sum_i F_i \beta_i \sigma_{Y_i}^0 \theta_{i_U}}{\sum_i F_i \sigma_{Y_i}^0}, \quad \max_i \frac{(P_U - P_L) E_i / l_i}{2\sigma_{Y_i}^0 \cdot \sum_j E_j F_j / l_j}\right\},$$

which represents the well-known fact that a self-equilibrated stress field caused by a constant (in time) temperature field does not affect the shakedown of structures subjected to variable loads but only if the temperature dependence of the material constants (in this case—the yield stresses) is disregarded ($\beta_i = 0$).

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